ON A THEOREM OF LUCAS

By M. B. Porter

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS

Received by the Academy, March 10, 1916

In a paper on the *Geometry of Polynomials*[1] Lucas has an interesting generalization of Rolle's theorem, to wit: *That the zeros of any polynomial $F'(z)$ lie inside any closed convex contour inside of which the zeros of $F(z)$ lie.*

Many proofs[2] of this theorem have been given, but no one seems to have pointed out that the theorem is applicable to integral transcendental functions of the type $I_0(z) = \Pi_1^\infty (1-z/\alpha_1)$ where $\Sigma_1^\infty |1/\alpha_i|$ is convergent, i.e., functions of zeroth order (*genre* zero).

We shall show that this theorem can be generalized so as to give information concerning the distribution of the zeros of the derivative of certain rational functions and certain transcendental functions of the type $I_0(z)/\bar{I}_0(z)$.

We begin by giving a very elementary proof (perhaps new) of Lucas' theorem.

Proof. Since, in the finite part of the complex plane $F'(z)/F(z) = \Sigma_1^n(z-\alpha_i)^{-1}$, where $\alpha_1, \quad . \quad . \quad . \quad , \alpha_n$ are the zeros of $F(z)$, can vanish only when $F'(z)$ vanishes, we have only to show that $\Sigma_1^n (z-\alpha_i)^{-1}$ can vanish only *inside* the convex contour mentioned. Now since the contour is convex, all the vectors $z-\alpha_i$ drawn from a point z *outside* the contour lie *inside* the arms of an angle *less* than 180°; the same thing will be true of the vectors $(z-\alpha_i)^{-1}$ (obtained by inverting and reflecting in the axis of reals through the point z). But such a set of vectors cannot form a closed[3] polygon, and hence the theorem is proved. It is now at once evident that the theorem is true for functions of the type $I_0(z)$ and we have, for example, a theorem of Laguerre's that: *If the zeros of $I_0(z)$ are all real, so are those of $I'_0(z)$.*

If $F(z) = P_n(z) / P_m(z)$, where P_n and P_m are polynomials or functions of the type $I_0(z)$ whose zeros α_i and β_i respectively lie inside of closed convex contours C_n and C_m which are *external* to each other, the proof given above shows that, if $\Phi(z) = P_n(z)/P_m(z)$, then

$$\frac{\Phi'(z)}{\Phi(z)} = \left(\log \frac{P_n(z)}{P_m(z)}\right)' \equiv \sum_1^n \frac{1}{z-\alpha_i} - \sum_1^m \frac{1}{z-\beta_i}$$

can have no zeros in the region swept out by such tangent straight lines to C_m as can be moved parallel to themselves into tangency with

C_n without cutting either C_m or C_n or passing through the position of the line infinity. This is a region which can be easily marked out on the complex plane and will have inside of it neither of the contours C_m and C_n.

Thus our theorem asserts that if the zeros of P_n are inside of one branch of an hyperbola and the zeros of P_m are inside the other branch, all the zeros of Φ' are *inside* of the hyperbola, or again, if all the zeros of P_n are real and lie in the interval (1) $x > a$, while all the zeros of P_m are real and lie in the interval (2) $x < b \leqq a$, then $\Phi'(z)$ has no complex zeros and all of its zeros lie in the intervals (1) and (2).

[1] *J. Ec. Polytech., Paris,* 28.

[2] All these proofs save one by Hayashi (*Annals of Mathematics,* March, 1914) are based on dynamical considerations. Fejér, Ueber die Wurzel vom kleinstein absoluten Betrage, etc., *Leipzig, Math. Ann.,* 65, 417, attributes the theorem to Gauss and gives a bibliography for it.

[3] If this is not at once intuitionally evident it can be shown by resolving the vectors in question into components parallel to the arms of the angle above mentioned.

INTERPRETATION OF THE SIMPLEST INTEGRAL INVARIANT OF PROJECTIVE GEOMETRY

By E. J. Wilczynski

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

Received by the Academy, March 13, 1916

If $y = f(x)$ is the cartesian equation of a plane curve, the integral

$$s = \int_{x0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{1}$$

which represents the length of the arc of this curve between the points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$, obviously remains unchanged when the curve is subjected to a plane motion. Therefore we may speak of s as an integral invariant of the group of motions, or as a *metric integral invariant.*

In the present paper we shall show how to find integrals connected with a given plane curve, whose values are not changed when the points of the plane are subjected to an arbitrary projective transformation. We shall speak of these integrals as *projective integral invariants.*

Let y_1, y_2, y_3, be the homogeneous coördinates of a point P_y, and let y_1, y_2, y_3 be given as linearly independent analytic functions of a parameter x. As x changes P_y will describe a non-rectilinear analytic curve C_y. There exists a uniquely determined linear homogeneous differential equation of the third order

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{2}$$